Euler's Formula

Math 123

March 2006

1 Purpose

Although Euler's Formula is relatively simple to memorize, it is actually a manifestation of a very deep mathematical phenomenon. In this activity, we will examine why Euler's Formula is true and try to come to grips with what it really means.

2 Materials

- Geofix/polydron shapes
- Paper
- Pencil

3 Vocabulary

- **edge**
- **Euler characteristic**
- **polyhedron**
- **tiling**
- vertex

4 Polygons

1. Begin with any polygon you like and consider its number of edges and vertices. What is the relationship?

2. Does this same relationship hold for any polygon?

3. What if we attach two polygons to each other? Try it. Counting the edge where they are attached, what is the relationship between number of vertices and number of edges? Does the same relationship hold for any two polygons we attach to each other?

4. What if we attach more polygons? Try making a few different shapes composed of a number of different polygons; is there a relationship between the number of vertices, the number of edges (including "interior" edges) and the number of polygons used to construct the shape? You might find the attached table useful to keep track.

5. If we use V to denote the number of vertices, E the number of edges and P the number of polygons in the shape, can you come up with a general formula in terms of V, E and P that holds for any shape you could make? Does the formula change if you ignore interior vertices and edges?

6. This formula, if it had a name, could be called Euler's Formula for flat shapes.

5 Why stay flat?

1. So far, all the shapes we've been investigating have been flat, but there's no reason not to take things out of the plane. Try taking one of the shapes you made and bending it so that it's no longer flat; does this affect the formula you came up with?

2. Try some more three-dimensional shapes...does the relationship still hold? Can figure out a way to make a shape where it doesn't hold?

6 Polyhedra

1. Try making a cube or a pyramid. Does the relationship you observed above still hold? Again, you may find the table useful to keep track of the numbers of vertices, edges and polygons.

2. Cubes and pyramids are both examples of polyhedra (singular: polyhedron); can you make a polyhedron with 4 faces (a tetrahedron)? 5 faces (pentahedron)? More? In each case, can you find a relationship between the numbers of vertices, edges and polygons used to make the polyhedron?

3. Can you express this relationship in a formula in terms of V, E and P that will hold for all polyhedra? Is this the same as the formula you found before?

4. This formula is called **Euler's Formula**, though in textbooks it is often written with an F (for "faces") instead of a P.

7 Why?

1. Euler's Formula seems like it holds for any polyhedron and textbooks assure us that it does, but we'd like to know *why* it's true. To that end, try removing a single face from one of the polyhedra you just made. How does this affect the number of vertices, edges and polygons in the resulting shape?

2. What's the relationship between V, E and P for the new shape you've just made? Does this look familiar?

3. Can you take this shape and, by bending and possibly disconnecting some

of the polygons (but not removing any), make it flat?

4. In making the shape flat, did you change the number of vertices, edges or polygons in it? Does this shed any insight into question 2?

8 What's the difference?

1. What's the difference between a polyhedron and a flat shape or some other three-dimensional shape like those you made in section 2?

2. It turns out that this is actually a very difficult mathematical question. Although we can **see** the difference between a polyhedron and another sort of shape pretty easily, it's rather difficult to pin the difference down in formulas. So difficult, in fact, that for hundreds of years mathematicians didn't even realize it was a problem. But that's exactly what we've just done! The two formulas you figured out above enable you to instantly tell the difference between a polyhedron and a non-polyhedron just by counting up the number of vertices, edges and faces and calculating $V - E + P$ (this number is called the **Euler characteristic** of a shape).

3. **Challenge Problem** Can you make a shape for which $V - E + P$ is not 1 or 2?

Shape	Vertices	Edges	Polygons